He will always carry on
Some things are lost, some things are found,
They will keep on speaking his name
Some things are changed, some still the same.[1]

To Alan Baker

APPLICATIONS OF BAKER THEORY TO THE CONJECTURE OF LEOPOLDT

PREDA MIHĂILESCU

Abstract. In this paper we give a short, elementary proof of the following too extreme cases of the Leopoldt conjecture: the case when $\mathbb{K}/\mathbb{Q}$ is a solvable extension and the case when it is a totally real extension in which p splits completely. The first proof uses Baker theory, the second class field theory. The methods used here are a sharpening of the ones presented at the SANT meeting in Göttingen, 2008 and exposed in [6], [7].

1. Introduction

Let $\mathbb{K}/\mathbb{Q}$ be a finite galois extension and p be a rational prime. It was conjected by Leopoldt in [5] that the p - adic regulator of $\mathbb{K}$ does not vanish. Some equivalent statements are explained below. The conjecture was proved for abelian extensions in 1967 by Brumer [2], using a local version of Baker's linear forms in logarithms: the result is known as the Baker-Brumer theorem. A theorem proved by Ax in [1] allows to relate the Leopoldt conjecture for abelian extensions to transcendency theory. In his paper, Ax mentions that he could expect his method to work also for non - abelian extensions. This was attempted by Emsalem and Kisilevski, who obtained in [3] results for some particular, non abelian extensions.

The main result of this paper is

[1]From a Hymn of *Pretenders*

Date: Version 1.0 October 21, 2009.
Key words and phrases. 11R23 Iwasawa Theory, 11R27 Units.

Theorem 1. *Let $\mathbb{K}/\mathbb{Q}$ be a galois extension and p an odd prime. The Leopoldt conjecture holds for $\mathbb{K}$ and p in the following cases:*

A *The group Gal $(\mathbb{K}/\mathbb{Q})$ is solvable.*
B *The extension $\mathbb{K}/\mathbb{Q}$ is totally real and splits p completely.*

We state from [2] the central theorem on p - adic forms in logarithms, which we shall use here:

Theorem 2 (Baker and Brumer)**.** *Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ and $\mathbb{U} \subset \overline{\mathbb{Q}}_p$ be the units. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be elements of $\mathbb{U}$ which are algebraic over $\mathbb{Q}$ and whose p - adic logarithms exist and are independent over $\mathbb{Q}$. These logarithms are then independent over $\mathbb{Q}'$, the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$.*

2. Baker theory and Leopoldt's conjecture

Let $\mathbb{K}/\mathbb{Q}$ be an arbitrary galois field with group G, let p be a rational prime and $P = \{\wp \subset \mathcal{O}(\mathbb{K}) \ : \ (p) \subset \wp\}$ be the set of conjugate prime ideals above p in $\mathbb{K}$.

We shall prove in this section two important consequences of the Theorem 2, one for absolute and one for relative galois extensions.

The algebra $\mathfrak{K}(\mathbb{K}) = \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is the product of all completions of $\mathbb{K}$ at the places in P:
$$\mathbb{K}_p = \prod_{\wp \in P} \mathbb{K}_\wp.$$
The global field $\mathbb{K}$ is dense in $\mathbb{K}_\wp$ in the product topology and G acts on this completion faithfully, so for any $x \in \mathbb{K}_p, x = \lim_n x_n, x_n \in \mathbb{K}$ and for all $g \in G$ we have $g(x) = \lim_n g(x_n)$. The field $\mathbb{K}$ is dense in $\mathfrak{K}$ (under the product of the topologies of $\mathbb{K}_\wp$), the units $U \subset \mathbb{K}_p$ are products of the units in $U_\wp \subset \mathbb{K}_\wp$ and E embeds diagonally to $\overline{E} \subset U$. We denote by $\iota_\wp$ the natural map $\mathfrak{K} \to \mathbb{K}_\wp$ and identify $a \in \mathbb{K}$ with its image in $\mathfrak{K}$. The one - units $U^{(1)}$ are defined naturally as the product $U^{(1)} = \prod_{\wp \in P} U^{(1)}_\wp$, with $U^{(1)}_\wp = \{x \in U_\wp : x - 1 \in \mathcal{M}_\wp\}$ and $\mathcal{M}_\wp$ the maximal ideal of $U_\wp$. Let $\mathbb{M}/\mathbb{K}$ be the product of all $\mathbb{Z}_p$ - extensions and $\Delta =$ Gal $(\mathbb{M}/\mathbb{K})$. The global Artin symbol is an homomorphism $\varphi : U^{(1)} \to \Delta$ of $\mathbb{Z}_p[G]$ - modules and there is an exact sequence

$$1 \to \overline{E} \to U^{(1)} \to \Delta \to 1. \tag{1}$$

We let $U' = \{x \in U^{(1)} : \mathbf{N}_{\mathbb{K}/\mathbb{Q}}(x) = 1\}$; then $\overline{E} \subset U'$ by definition. Therefore $U^{(1)}/U' \cong U^{(1)} \cap \mathbb{Q}_p = U^{(1)}(\mathbb{Q}_p)$ is a quotient which is fixed by G. It is known that $\widetilde{U^{(1)}} \cong \mathbb{Q}_p[G]$, so Δ is a quasi-cyclic $\mathbb{Z}_p[G]$ (see also [8]). Since $\mathbb{K}_\infty/\mathbb{K})$ is a $\mathbb{Z}_p$ - extension whose galois group

is a quotient of Δ, invariant under G; but Δ being $\mathbb{Z}_p[G]$ - cyclic, it follows that this quotient is unique up to quasi-isomorphism. We let $\Delta' =$ Gal $(\mathbb{M}/\mathbb{K}_\infty)$; we then have the additional exact sequence

$$1 \to \overline{E} \to U' \to \Delta' \to 1. \tag{2}$$

We refer to [8], §§2.1, 2.2 and 3.1 for more details on Minkowski units, idempotents of non commutative group rings and the associated annihilators, supports and components of $\mathbb{Z}_p[G]$ - modules. We also refer to §2.3 for the description of a choice of the base field $\mathbb{K}$, which contains the $p^\kappa-$th roots of unity and has some pleasant properties, such as the fact that the p - ranks of all Λ - modules of finite rank are stationary, all ideals that capitulate have order bounded by p^κ and $v_p(|G|) \leq \kappa$. In the same section we describe Weierstrass modules – which are $\mathbb{Z}_p$ - torsion free, infinite Λ - modules of finite p - rank – and prove the fundamental formula

$$\mathrm{ord}(a_n) = p^{n+1+z(a)} \quad \forall n > 0,$$

which characterizes the orders of $a = (a_n)_{n\in\mathbb{N}} \in W \subset A$, when W is Weierstrass. Here $\mathbb{Z} \ni z(a) \leq \kappa$ is a constant depending on a but not on a_n. We use the notation $\varsigma(x) = x^{p^\kappa}$ for x in an abelian group; the choice of ς is such that $\varsigma(A)$ is a Weierstrass module and for $a \in \underline{A}$, the finite p - torsion part of A, we have $\varsigma(a) = 1$. We write $\mathbb{H}, \Omega$ for the maximal p - abelian, unramified, respectively p - ramified extensions of $\mathbb{K}_\infty$. If $\mathbb{F}/\mathbb{K}_\infty$ is any extensions and $F_0 =$ Gal $(\mathbb{F}/\mathbb{K}_\infty)^\circ$ is the $\mathbb{Z}_p$ - torsion of its galois group, we write $\overline{\mathbb{F}} = \mathbb{F}^{F_0}$: an extension which is either trivial or has a Weierstrass - module as galois group; this group may still be a free Λ - module.

The conjecture of Leopoldt says that

$$\mathbb{Z}_p\text{-rk}(\overline{E}) = \mathbb{Z}\text{-rk}(E).$$

Let $\delta \in E$ be a Minkowski unit with $\delta \equiv 1 \bmod p^2$. Then the p - adic logarithms of δ^g exist in all completions $\mathbb{K}_\wp$ and for all $g \in G$. If $A \subset \mathbb{K}_p$ is a multiplicative group, we write the action of G exponentially, so $a^g = g(a)$. If G is not commutative and $g, h \in G$ we have

$$a^{gh} = (a^g)^h = h \circ g(a), \tag{3}$$

and the definition of a contravariant multiplication $G \times G \to G$ with $g \cdot h = h \circ g$ makes A into a right $\mathbb{Z}_p[G]$ - module, and likewise for $\mathbb{Z}[G]$ - modules. In particular, $U, \overline{E}$ and are $\mathbb{Z}_p[G]$ - modules and Minkowski units generate submodules of maximal $\mathbb{Z}_p$ - rank: since $\mathbb{K}$ is dense in $\mathbb{K}_p$, it follows that $\mathbb{Z}_p$-rk$(\overline{E}) = \mathbb{Z}_p$-rk$(\delta^{\mathbb{Z}_p[G]})$. With this structure we

also define

$$\delta^{\top} = \{x \in \mathbb{Z}[G] \ : \ \delta^x = 1\}, \quad \delta_p^{\top} = \{x \in \mathbb{Z}_p[G] \ : \ \delta^x = 1\},$$

the $\mathbb{Z}$ - and $\mathbb{Z}_p$ annihilators of δ. Then Leopoldt's conjecture is also equivalent to

$$\delta_p^{\top} = \delta^{\top} \otimes_{\mathbb{Z}} \mathbb{Z}_p. \tag{4}$$

In the context of this conjecture we are interested in ranks and not in torsion of modules over rings. It it thus a useful simplification to tensor these modules with fields, so we introduce the following

Definition 1. *Let G be a finite group and A, B a $\mathbb{Z}$, respectively a $\mathbb{Z}_p$ - module, which are torsion free. Let $a \in A, b \in B$. We denote*

$$\begin{aligned} \hat{A} &= A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \hat{a} = a \otimes 1, \\ \widetilde{B} &= B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad \widetilde{b} = a \otimes 1, \end{aligned}$$

Note that $\mathbb{Z}$-rk$(A) = \mathbb{Q}$-rk$(\hat{A})$ and $\mathbb{Z}_p$-rk$(B) = \mathbb{Q}_p$-rk$(\widetilde{B})$. We shall simply write rank (X) *for the rank of a module when the ring of definition is clear (being one of $\mathbb{Z}, \mathbb{Z}_p$ or $\mathbb{Q}, \mathbb{Q}_p$.)*

For instance, $\widetilde{E} = \overline{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The definition of $\hat{E}$ is not important for absolute extensions, but relevant in relative extensions $\mathbb{L}/\mathbb{K}$, when $\mathbf{N}_{\mathbb{L}/\mathbb{K}}(E(\mathbb{L})) \subsetneq E(\mathbb{K})$.

We start with the case of an absolute extension $\mathbb{K}/\mathbb{Q}$, as introduced above. Let $r = r_1 + r_2 - 1 = \mathbb{Z}\text{-rk}(E)$ and $H = \{g_1, g_2, \ldots, g_r\} \subset G \setminus \{1\}$ be a maximal set of automorphisms, such that δ^{g_i} are $\mathbb{Z}$ - independent. In particular, there is a $\mathbb{Z}$ - linear map $e : \mathbb{Z}[G] \to \mathbb{Z}[H]$ such that

$$\delta^{\sigma} = \delta^{e(\sigma)} \tag{5}$$

for each $\sigma \in G$. The map is the identity on H and extends to G due to the Minkowski property, which implies that $\delta^{\mathbb{Z}[H]} = \delta^{\mathbb{Z}[G]}$.

We have the following consequence of Theorem 2

Lemma 1. *Let the notations be like above and $\mathbb{Z}' = \mathbb{Q}' \cap \mathbb{Z}_p$ be the integers in the algebraic closure $\mathbb{Q}' \subset \mathbb{Q}_p$ of $\mathbb{Q}$. Then*

$$\delta_p^{\top} \cap \mathbb{Z}'[G] = \delta^{\top}.$$

In particular, if $\delta_p^{\top} = \alpha \mathbb{Z}_p[G]$ with $\alpha \in \mathbb{Z}'[G]$, then Leopoldt's conjecture holds for $\mathbb{K}$.

Proof. Let $\wp \in P$ be fixed and $\delta_\tau = \iota_\wp(\delta^\tau)$; then $\delta_\tau \in \mathbb{Z}'$. Since $\{\delta^\tau \ : \ \tau \in H\}$ are $\mathbb{Z}$ - independent, $\{\delta_\tau \ : \ \tau \in H\}$ are a fortiori $\mathbb{Z}$ - independent. Indeed, if $t \in \mathbb{Z}[H]$ was a linear dependence for δ_τ, such that $\iota_\wp(\delta^t) = 1$, then $d = \delta^t \in E$ verifies $\iota_\wp(d) = 1$. But in the diagonal

embedding of E, a projection is 1 if and only if the unit itself is 1, thus $d = 1$: a contradiction of the independence of $\delta^{\tau}, \tau \in H$.

Let $\theta_0 \in \delta_p^{\top} \cap \mathbb{Z}'[G]$; in view of (5), $\theta = e(\theta_0) \in \delta_p^{\top} \cap \mathbb{Z}'[H]$ is also an annihilator. Let $\theta = \sum_{\tau \in H} c_{\tau}\tau,\ c_{\tau} \in \mathbb{Z}'$. We show that Theorem 2 implies $\theta = 0$, so $\theta_0 \in e^{-1}(0) \subset \mathbb{Z}[G]$ for all $\theta_0 \in \delta_p^{\top} \cap \mathbb{Z}'[G]$, which is the claim.

We have $\iota_{\wp}(\delta^{\theta}) = \prod_{\tau \in H} \delta_{\tau}^{c_{\tau}} = 1 \in \mathbb{K}_{\wp}$, and taking the p - adic logarithm we find the vanishing linear form in logarithms

$$\sum_{\tau \in H} c_{\tau} \log_p(\delta_{\tau}) = 0.$$

Since $c_{\tau}, \delta_{\tau} \in \mathbb{Z}'$ and $\{\delta_{\tau} : \tau \in H\}$ are $\mathbb{Z}$ - independent, the Theorem of Baker and Brumer implies that $\theta = 0$.

Consequently, if $\delta_p^{\top} = \theta_0 \mathbb{Z}_p[G]$ and $\theta_0 \in \mathbb{Z}'[G]$, then the proof above shows that $\theta_0 \in \mathbb{Z}[G]$, which implies (4) and confirms Leopoldt's conjecture. $\square$

The following definition brings relative annihilators into focus.

Definition 2. *Let $\mathbb{L} \supset \mathbb{K}$ be an extension of number fields with the following properties:*

1. *$\mathbb{L}/\mathbb{Q}$ is a galois extension with group G and $H =$ Gal $(\mathbb{L}/\mathbb{K})$.*
2. *Let the relative annihilator of $e \in E(\mathbb{L})$ be defined by*

$$\begin{aligned} \widetilde{e}_{\mathbb{L}/\mathbb{K}}^{\top} &= \{x \in \mathbb{Q}_p[H] : \widetilde{e}^{x} \in \widetilde{E(\mathbb{K})}\}, \\ e_{\mathbb{L}/\mathbb{K}}^{\top} &= \widetilde{e}_{\mathbb{L}/\mathbb{K}}^{\top} \cap \mathbb{Z}_p[H]. \end{aligned}$$

Then for any global Minkowski unit $\delta \in E(\mathbb{L})$ we have

$$\widetilde{\delta}_{\mathbb{L}/\mathbb{K}}^{\top} = \mathbf{N}_{\mathbb{L}/\mathbb{K}} \cdot \mathbb{Q}_p[H].$$

If points 1. and 2. hold for $\mathbb{L}/\mathbb{K}$, we say that $\mathbb{L}/\mathbb{K}$ is relative Leopoldt extension, or rL - extension. If in addition $\mathbb{L}$ is real, then the extension is a real relative Leopoldt, or RL.

Remark 1. *Bruno Anglès observed in connection with earlier attempts to use Baker theory, that these attempts suggest an approach using relative extensions, maybe even a relative Leopoldt conjecture, stating that if $\mathbb{L}/\mathbb{K}$ is RL and Leopoldt's conjecture holds for $\mathbb{K}$, then it holds for $\mathbb{L}$. The use of relative annihilators leads to a proof of statement A. in Theorem 1. However the relative conjecture encouters a severe obstruction due to the fact that Baker theory only allows statements on the relative annihilator in $\mathbb{Q}_p[H]$, but the relative conjecture requires annihilators in $\mathbb{Q}_p[G]$.*

We consider next the case of relative abelian extensions:

Lemma 2. *Abelian extensions* $\mathbb{L}/\mathbb{K}$ *with* $\mathbb{L}/\mathbb{Q}$ *galois are relative Leopoldt extensions. Furthermore, if* $\mathbb{L}/\mathbb{K}$ *is galois such that* $\mathbb{L}_\wp/\mathbb{K}_\wp$ *is abelian for all prime ideals* $\wp \in P$*, then* $\mathbb{L}_\wp/\mathbb{K}_\wp$ *is a local relative Leopoldt extension with respect to* $\iota_\wp(e)$ *for all Minkowski units* $e \in E(\mathbb{L})$.

Proof. Let $H =$ Gal $(\mathbb{L}/\mathbb{K})$ be abelian; the extension $\mathbb{L}/\mathbb{K}$ arises from a succession of cyclic extensions of prime degree, so it suffices to assume this case. Let $H = \langle \sigma \rangle$ with $|H| = [\mathbb{L} : \mathbb{K}] = q$, for a prime q which is not necessarily different from p. The group $\mathbb{Q}_p[H]$ decomposes as $\mathbb{Q}_p[H] = e_1\mathbb{Q}_p[H] \oplus (1 - e_1)\mathbb{Q}_p[H]$, where e_1 is the idempotent $\frac{N}{q}$, with $N = \mathbf{N}_{\mathbb{L}/\mathbb{K}}$. Suppose thus that $\widetilde{\delta}^{\top}_{\mathbb{L}/\mathbb{K}} = (ae_1 + be_\chi)\mathbb{Q}_p[H]$, where e_χ is a (non trivial) sum of central idempotents for the augmentation part $\mathbb{Q}_p[I_{\mathbb{L}/\mathbb{K}}]$ and $a, b \in \{0, 1\}$. We shall show that $a = 1$ and $b = 0$.

From the definition of $\widetilde{\delta}^{\top}_{\mathbb{L}/\mathbb{K}}$ we have

$$\widetilde{\delta}^{ae_1+be_2} = N(\widetilde{\delta})^a \cdot \widetilde{\delta}^{be_2} \in \widetilde{E}(\mathbb{K}).$$

Since $N(\widetilde{\delta}) \in \widetilde{E}(\mathbb{K})$, we also have $d := \widetilde{\delta}^{be_2} \in \widetilde{E}(\mathbb{K})$. The group H is cyclic and e_2 is in the augmentation, so $e_2N = 0$. Taking the norm in the definition of d and using the fact that $d^\sigma = d$ and thus $N(d) = d^q$, we find that

$$\widetilde{\delta}^{be_2N} = d^q = \widetilde{\delta}^{be_2q} = 1.$$

But $e_2q \in \mathbb{Z}_p[H]$ and thus $\delta^{be_2q} = 1$: starting from a relative relation we deduced an absolute annihilator of δ which is algebraic. We may apply the Lemma 1, concluding that $e_2 = 0$, since by hypothesis there is no rational dependence for δ in the augmentation. This completes the proof. $\square$

As a consequence, we have

Lemma 3. *Solvable extensions* $\mathbb{L}/\mathbb{K}$ *with* $\mathbb{L}/\mathbb{Q}$ *real and galois are RL - extensions.*

Proof. Since H is solvable, there is a chain of intermediate extensions $\mathbb{K}_0 = \mathbb{K} \subset \mathbb{K}_1 \subset \mathbb{K}_2 \subset \ldots \subset \mathbb{K}_r = \mathbb{L}$ such that $\mathbb{K}_{i+1}/\mathbb{K}_i$ is abelian for $i = 0, 1, \ldots, r-1$ and $\mathbb{L}/\mathbb{K}_i$ is solvable for all i. The Lemma 2 holds for all $\mathbb{K}_{i+1}/\mathbb{K}_i$. Let $N_i = \sum_{\sigma \in \mathrm{Gal}(\mathbb{K}_{i+1}/\mathbb{K}_i)} \sigma$; then $N = N_0 \circ N_1 \circ \ldots \circ N_{r-1}$. The claim follows by induction and we illustrate this for the case $r = 2$, so Gal $(\mathbb{L}/\mathbb{K}_1) = H_1$, Gal $(\mathbb{K}_1/\mathbb{K}) = H_0$ and $H = H_0 \ltimes H_1$. Furthermore, $\mathbb{Q}_p[H] = \mathbb{Q}_p[H_0] \ltimes \mathbb{Q}_p[H_1]$ where the semidirect product $a_0 \ltimes a_1$, with $a_i \in \mathbb{Q}_p[H_i], i = 0, 1$ is defined term-wise; $N = N_0 \ltimes N_1$ follows from this definition.

We know from the lemma that $\widetilde{\delta}^{\top}_{\mathbb{L}/\mathbb{K}_1} = N_1\mathbb{Q}_p[H_1]$ and letting $\delta_1 = N_1(\delta) \in \mathbb{K}_1$, the same lemma yields $\widetilde{\delta_1}^{\top}_{\mathbb{K}_1/\mathbb{K}} = N_0\mathbb{Q}_p[H_0]$. It follows that $\widetilde{\delta}^{\top}_{\mathbb{L}/\mathbb{K}} \subset \widetilde{\delta_1}^{\top}_{\mathbb{K}_1/\mathbb{K}} \ltimes N_1\mathbb{Q}_p[H_1] = N_0\mathbb{Q}_p[H_0] \ltimes N_1\mathbb{Q}_p[H_1] = N\mathbb{Q}_p[H]$. This way we may prove inductively that $\mathbb{L}/\mathbb{K}_i$ is RL for $i = r-2, r-3, \ldots, 0$. □

The last lemma leads to:

Corollary 1. *Point A. in Theorem 1 is true.*

Proof. Suppose that $\mathbb{L}/\mathbb{Q}$ is a real extension with solvable group H. If $\delta \in E(\mathbb{L})$ is a Minkowski unit, then Lemma 3 implies that its relative annihilator $\widetilde{\delta}^{\top}_{\mathbb{L}/\mathbb{Q}} = \mathbf{N}_{\mathbb{L}/\mathbb{Q}}\mathbb{Q}_p[H]$. Since the base field is $\mathbb{Q}$, the relative annihilator is equal to the absolute one and it follows that $\mathbb{Z}_p\text{-rk}(\overline{E}) = \mathbb{Z}\text{-rk}(E)$ and Leopoldt's conjecture is true.

The restriction that $\mathbb{L}$ be real is not important. If $\mathbb{L}$ is complex, with solvable group, then $< \jmath >=$ Gal $(\mathbb{L}/\mathbb{L}^+) \subset H$ is a normal subgroup, so $\mathbb{K}^+/\mathbb{Q}$ is galois, and Leopoldt's conjecture holds in this case too. □

We are prepared to prove

Theorem 3. *Let $\mathbb{K}/\mathbb{Q}$ be a totally real extension with group G and $\mathbb{M}/\mathbb{K}$ be the product of all $\mathbb{Z}_p$ extensions of $\mathbb{K}$, $\mathbb{K}_\infty$ the cyclotomic $\mathbb{Z}_p$ - extension of $\mathbb{K}$ and $\mathbb{H}/\mathbb{K}_\infty$ be the maximal p - abelian p - unramified extension. Then $\mathbb{H} \cap \mathbb{M} = \mathbb{K}_\infty$.*

Proof. We adopt a class field theoretic approach for our proof. Let $\mathbb{K}$ be like in the hypothesis and $(\mathbb{K}_n)_{n\in\mathbb{N}}$ be the intermediate fields of its cyclotomic $\mathbb{Z}_p$ - extension. We assume that $\mathbb{K}$ is such that the Leopoldt defect $\mathcal{D}(\mathbb{K}_n) = \mathcal{D}(\mathbb{K})$ is constant for all $n \geq 0$ and let $\mathbb{L} = \mathbb{K}[\zeta_p]$.

Firstly, we note that we can exclude the case that $\mathbb{M}/\mathbb{K}_\infty$ contains subextensions which split the primes above p: this follows by using the Iwasawa skew symmetric pairing and was proved in [8], Theorem 3 of §3.3. It remains that $\mathbb{M} \subset \Omega_E$ and Gal $(\mathbb{M}/\mathbb{K}_\infty) \hookrightarrow \mathbf{B}$, where $\mathbf{B} \subset A$ is the submodule generated by classes containing ramified primes above p, as defined in [8].

Let $E_n = E(\mathbb{K}_n)$ and $U'_n = U'_n(\mathbb{K}_n)$; we define $U'_\infty = \cup_n U'_n$ and $E_\infty = \cup_n E_n$. Then it is known that U'_∞/E_∞ is a torsion Λ - module and thus, by choice of ς, we obtain a Weierstrass module $\varsigma(U'_\infty/E_\infty)$; since $\mathbb{K}$ is totally real, by reflection we see that

$$\varsigma(U'_\infty/E_\infty)^\bullet \hookrightarrow A^-(\mathbb{L}).$$

Let $X''_n = \{x \in X_n : N_{\mathbb{K}_n,\mathbb{K}}(x) = 1, n > \kappa\}$ for $X \in \{U'_n, E_n\}$. We let $W_n = \varsigma(U''_n/\overline{E}''_n)$ and $W = \varsigma(U''_\infty/\overline{E}''_\infty)$. Then W is a Weierstrass

module and we denote by F its characteristic polynomial. Since the T - part of W is trivial, $T \nmid F(T)$. The Leopoldt defect is stationarity, so $\widetilde{E}''_n = (\widetilde{U}''_n)^+$. Applying F annihilates the diverging part in W_n and we obtain:

$$\left[\left((U''_n)^+\right)^{F(T)} : (E''_n)^{F(T)}\right] < M, \tag{6}$$

for a fixed upper bound M. In particular, since $(T, F(T)) = 1$, there is a fixed $m \geq \kappa$, depending on $F(T)$ and $|G|$, such that for all $n > 0$ we have

$$((U''_n)^+)^{p^m} \subset E''_n \cdot U''^T_n. \tag{7}$$

Note that $E''_n \cap U''^T_n = E''^T_n$; indeed, let $e \in E''_n \cap U''^T_n$. By Hilbert 90 and the choice of e, there are a $w \in \mathbb{K}_n^\times$ and $\xi \in U''_n$ with $e = w^T = \xi^T$, and it follows that $w = w_0 \cdot \xi$ for some $w_0 \in U(\mathbb{K})$. The ideal (w) is ambig above $\mathbb{K}$; since p^κ annihilates the class group A_0, we have $(w^{p^\kappa}) = (\pi \cdot \gamma)$, with $\gamma \in \mathbb{K}, (\gamma, p) = 1$ and π a product of ideals above p. There is a unit $e_1 \in E_n$ such that $e_1 \pi \gamma = w^{p^\kappa} = (w_0 \xi)^{p^\kappa}$ and since ξ is a local unit, it follows that $\pi | w_0$ or $\pi = 1$; moreover, $(\pi\gamma)^T = \gamma^T = 1$. It remains that $e^{p^\kappa} = w^{Tp^\kappa} = e_1^T$, so $e \in E_n^T \cap E''_n$. Finally we show that we may choose $e_1 \in E''_n$. From $(\xi/e_1)^T = 1$ we have $e_1 = c\xi, c \in \mathbb{K}$ and $N_{n,0}(e_1) = c^{p^{n-\kappa}}$, so $c \in E(\mathbb{K})$; the unit $e_2 := e_1/c \in E''_n$ verifies $e_2^T = e$, which confirms this claim.

In particular, for sufficiently large n, we have

$$\begin{aligned} p\text{-rk}\left(E''_n / \left(E''^{p^n}_n \cdot (U''^T_n \cap E''_n)\right)\right) &= p\text{-rk}\left(E''_n / E''^{(p^n,T)}_n\right) \\ &= r_2 - 1. \end{aligned} \tag{8}$$

Assume that $\mathbf{B}$ is infinite, and let $\alpha^\top \in \mathbb{Q}_p[G]$ be its canonic annihilator[1]. We shall write $\mathbb{Z}_p\text{-rk}(\varsigma(\mathbf{B})) = \mathcal{D}(\mathbb{K})$: this is the case for the totally real extensions $\mathbb{K}$ which split the primes above p, as we show in the corollary below; note however that the claim of this proposition is more general and holds independently of this assumption. It does follow from the general case of Leopoldt's conjecture too.

We shall use (7) and (8) for proving that $\mathcal{D}(\mathbb{K}) = 0$. The core observation is that, if $\mathcal{D}(\mathbb{K}) > 0$, then there is a defect emerging in the $\mathbb{Z}$ - rank of E''_n, which raises a contradiction to (8).

For $\wp \in P$, we let $\wp_n \in A_n$ be the primes above $\wp$ and $a_n = [\wp_n] \in A_n$ be their classes, with diverging orders $\text{ord}(a_n) = p^{n+1+z(a)}$. If α_n approximates $|G|\alpha$ to the power $p^{n+\kappa+1}$, say, then there is a $\nu_n \in \mathbb{K}_n$ such that $(\nu_n) = \wp_n^{p^\kappa \cdot \alpha_n}$ and $\nu_n^T = e_0 \in E''_n$. Since $\nu_n^{|G|(1-\alpha_n)} \in (\mathbb{K}_n^\times)^{p^n}$,

[1] see [8] §7.1 for a definition of canonic annihilators for cyclic modules over non - abelian group rings

it follows that the unit e_0 is annihilated in $E''_n/E''^{(T,p^n)}_n$ by $|G|(1-\alpha_n)$. We shall show that p-rk $\left(E''_n/E''^{T,p^n}_n\right) = r_2 - 1 - \mathcal{D}(\mathbb{K})$, so (8) implies $\mathcal{D}(\mathbb{K}) = 0$.

Let $B \subset \mathbf{B}^\top$ be an irreducible elementary module with $\widetilde{B}^\top = \beta \in \mathbb{Q}_p[G]$, an idempotent dividing α, and let β_n be rational approximants of $|G| \cdot \beta$. Suppose that there is a unit $e \in E''_n$ with $e^{\beta_n} \in \nu_n^{T\mathbb{Z}[G]}$; since $N_n(e) = 1$, it follows from [8], Lemma 16, that $\varsigma(e) = \pi^T$ for some p - unit π, so there is a $\theta \in \mathbb{Z}[G]$, with $(\pi) = \wp_n^\theta$. We may write

$$|G|\theta \equiv c\alpha_n + b(1-\alpha_n) \mod p^n\mathbb{Z}[G]; \; c, b \in p^\kappa\mathbb{Z}[G],$$

and claim that $b \equiv 0$ modulo a large power of p. Upon multiplication with $|G|(1-\alpha_n)$ we obtain a unit $e_1 = e^{|G|(1-\alpha_n)} = \pi_1^T$ with $(\pi_1) = \wp_n^{b|G|(1-\alpha_n)+O(p^n)}$. The identity requires that the ideal $\wp_n^{b|G|(1-\alpha_n)}$ be principal and since α is the minimal annihilator of a, for large n, this implies $b \equiv 0 \bmod p^{n-(m+\kappa)}$, which was our claim on b. It follows that $\beta_n \in \alpha_n\mathbb{Z}[G] + p^{n-(m+\kappa)}\mathbb{Z}[G]$. But then, for $m' = 2(m+\kappa)$ and $n > m'$, the quotient $(E''_n)/((E''_n)^T \cdot (E''_n)^{p^{n-m'}})$ has p - rank $r_2 - 1 - \mathcal{D}(\mathbb{K})$ and (8) implies $\mathcal{D}(\mathbb{K}) = 0$, which completes this proof. □

As a consequence,

Corollary 2. *Point B. in Theorem 1 is true.*

Proof. Assume that $\mathbb{K}/\mathbb{Q}$ is totally real and splits p completely and let $\wp \in P$ be any prime above p. Then $\mathbb{K}_\wp = \mathbb{Q}_p$ and $\mathbb{M}_\wp/\mathbb{K}_\wp$ is in the product of the two $\mathbb{Z}_p$ - extensions of $\mathbb{Q}_p$: the unramified and the cyclotomic. Consequently $\mathbb{M}/\mathbb{K}_\infty$ must be unramified at p. This holds for all primes above p, so $\mathbb{M}/\mathbb{K}_\infty$ is totally unramified. However, by Theorem 3 we know that $\mathbb{M} \cap \mathbb{H} = \mathbb{K}_\infty$, so we must have $\mathbb{M} = \mathbb{K}_\infty$ and the Leopoldt conjecture holds in this case.

The same argument was used by Greenberg in [4] for showing that λ may take arbitrarily large values in abelian extenions $\mathbb{K}/\mathbb{Q}$. □

Remark 2. *It has been believed for a longer time that the two extreme cases treated by Theorem 1 are the easire one for Lepoldt's conjecture, so this short proof only confirms this general belief. The general proof, given in* [8]*, requires deeper class field theory.*

The obstruction encountered, when trying to generalize the results of this paper is the following: let $\mathbb{K}/\mathbb{Q}$ have group G and $\wp \in P$. The facts proven on relative annihilators imply quite easily that $\iota_\wp(\overline{E} \cdot \mathbb{Q}_p = \mathbb{K}_\wp$. Let $\Delta_\wp \subset \Delta =$ Gal $(\mathbb{M}/\mathbb{K})$; Leopoldt's conjecture would follow from Theorem 3, if we can prove that

$$\Delta_\wp \cong U_\wp^{(1)}/\iota_\wp(\overline{E}),$$

which may appear as a 'reasonable' localization of (1). It needs however not be true and all we can say is the following: if $D \subset G$ is the decomposition group of $\wp$, $\widetilde{\delta} = \widetilde{\xi}^{\alpha}$ and $\alpha = \sum_{\tau \in C} c_\tau \tau$ with $c_\tau \in \mathbb{Q}_p[D]$ and $C = G/D$, then $\sum_{\tau \in C} c_\tau \in \mathbf{N}_{\mathbb{M}_\wp/\mathbb{K}_\wp}$.

Acknowledgments: Much of the material presented here was completed after a two day visit of intensive work at the Laboratoire de Mathématique Nicolas Oresme of the University of Caen. I am most grateful to Bruno Anglés and David Vauclair for the helfpul and stimulating discussions which had an important contribution for clarifying the central ideas of these two papers.

References

[1] J. Ax. On the units of an algebraic number field. *Illinois Journal of Mathematics*, 9:584–589, 1965.
[2] A. Brumer. On the units of algebraic number fields. *Mathematika*, 14:121–124, 1967.
[3] M. Emsalem, H. Kisilevsky, and D. Wales. Indépendance linéaire sur $\overline{\mathbb{Q}}$ de logarithmes p - adiques de nombres algébriques et rang p - adique du groupe des unités d'un corps de nombres. *Journal of Number Theory*, 19:384–391, 1984.
[4] R. Greenberg. On the iwasawa invariants of totally real fields. *American Journal of Mathematics*, 98:263–284, 1973.
[5] H.-W. Leopoldt. Zur Arithmetik in abelschen Zahlkörpern. *J. Reine Angew. Mathematik*, 209:54–71, 1962.
[6] P. Mihăilescu. Leopoldt's conjecture for some galois extensions. In *Proceedings SANT*. Universitätsverlag Göttingen, 2009.
[7] P. Mihăilescu. On Leopoldt's conjecture and a special case of Greenberg's conjecture. In *Proceedings SANT*. Universitätsverlag Göttingen, 2009.
[8] P. Mihăilescu. The T and T^* components of Λ - modules and Leopoldt's conjecture. arxiv.org/abs/0905.1274, September 2009.

(P. Mihăilescu) Mathematisches Institut der Universität Göttingen
E-mail address, P. Mihăilescu: `preda@uni-math.gwdg.de`